JOURNAL OF COMPUTATIONAL PHYSICS 55, 302–312 (1984)

On the Gottlieb–Turkel Time Filter for Chebyshev Spectral Methods

SCOTT R. FULTON

Department of Atmospheric Science, Colorado State University, Fort Collins, Colorado 80523

AND

G. D. TAYLOR

Department of Mathematics, Colorado State University, Fort Collins, Colorado 80523

Received May 3, 1983

Highly accurate solutions to hyperbolic boundary value problems may be obtained using Chebyshev spectral methods. However, with explicit time differencing the increased resolution of the Chebyshev series near the boundaries necessitates using very small time steps for stability, thus compromising efficiency. Recently, a method of time filtering was proposed which is claimed to make explicit time integrations unconditionally stable so that time steps may be chosen by accuracy requirements alone. An analysis of this method shows that the filtering can in fact lead to absolute instability for any time step and that it does not relax the stability condition of the unfiltered method in a useful manner.

1. INTRODUCTION

Spectral methods may be used to obtain highly accurate solutions to hyperbolic partial differential equations [1]. For finite domains with general (nonperiodic) boundary conditions, Chebyshev series expansions are appropriate. Since the order of convergence is exponential for problems with infinitely differentiable solutions, the number of polynomials needed is small compared to the number of gridpoints needed for similar accuracy with finite difference methods.

However, the overall efficiency of Chebyshev spectral methods depends on the time differencing used. Conventional explicit schemes are easy to formulate, but require extremely small time steps for stability. Implicit and semi-implicit schemes allow much larger time steps, but are considerably more complicated to implement. Recently, a time filtering procedure to make explicit schemes for Chebyshev methods unconditionally stable has been proposed [2–4]. In this paper, we investigate the properties and usefulness of this procedure.

In Section 2 we illustrate Chebyshev spectral methods by formulating them for a

0021-9991/84 $3.00

simple model problem, the one-dimensional linear advection equation. The time filtering procedure is introduced in Section 3 and its effects on stability for the model problem are examined in Section 4. Concluding remarks are presented in Section 5.

2. CHEBYSHEV SPECTRAL METHODS

To illustrate Chebyshev spectral methods, consider the model problem

$$\begin{aligned} &\frac{\partial u}{\partial t}+\frac{\partial u}{\partial x}=0 \qquad (-1<x\leqslant 1, t>0),\\ &u(-1,t)=0 \qquad (t>0), \end{aligned} \tag{2.1}$$

where the initial condition $u(x, 0)$ is specified. The analytical solution u of (2.1) is simply

$$u(x,t)=\begin{cases} u(x-t,0) & x-t\geqslant -1 \\ 0 & x-t<-1 \end{cases}. \tag{2.2}$$

One seeks an approximate solution v of the form

$$v(x,t)=\sum_{n=0}^{N} \hat{v}_n(t)\, T_n(x), \tag{2.3}$$

where T_n is the Chebyshev polynomial of degree n, defined by $T_n(\cos\theta)=\cos(n\theta)$. There are two common ways of defining v, yielding slightly different approximations. In the tau method v is represented by its spectral coefficients $\hat{v}_n$ which are defined by

$$\left.\begin{aligned} &\frac{d\hat{v}_n}{dt}+\hat{v}_n^{(1)}=0 \qquad (n=0,\ldots,N-1),\\ &\sum_{n=0}^{N}(-1)^n\,\hat{v}_n=0. \end{aligned}\right\} \tag{2.4}$$

Here the $\hat{v}_n^{(1)}$ are the spectral coefficients of $\partial v/\partial x$, defined by

$$\hat{v}_n^{(1)}=\frac{2}{\pi c_n}\int_{-1}^{1}\frac{v_x(x,t)\,T_n(x)}{(1-x^2)^{1/2}}\,dx=\frac{2}{c_n}\sum_{\substack{p=n+1\\ n+p\ \mathrm{odd}}}^{N} p\hat{v}_p, \tag{2.5}$$

where $c_0=2$ and $c_1=\cdots=c_N=1$.

In contrast, in the collocation method (often referred to as the pseudospectral method) v is represented by its values v_j at the collocation points $x_j = \cos(j\pi/N)$ $(j = 0,..., N)$. The values v_j are defined by

$$\begin{aligned} \frac{dv_j}{dt} + v_j^{(1)} = 0 \qquad (j = 0,..., N-1), \\ v_N = 0, \end{aligned} \tag{2.6}$$

where $v_j^{(1)}(t) = v_x(x_j, t)$. This derivative may be computed by transforming the v_j to spectral space via

$$\hat{v}_n = \frac{2}{N\bar{c}_n} \sum_{j=0}^{N} \frac{1}{\bar{c}_j} v_j T_n(x_j), \tag{2.7}$$

where $\bar{c}_0 = \bar{c}_N = 2$ and $\bar{c}_1 = \cdots = \bar{c}_{N-1} = 1$, computing the derivative in spectral space using (2.5), and transforming back to physical space via

$$v_j^{(1)} = \sum_{n=0}^{N} \hat{v}_n^{(1)} T_n(x_j). \tag{2.8}$$

[Note that the spectral coefficients $\hat{v}_n$ obtained from (2.7) in the collocation method will differ from those defined by (2.4) in the tau method.] The semi-discrete system for either method may be written in matrix form as

$$\frac{d\mathbf{v}}{dt} = L_N \mathbf{v} \tag{2.9}$$

by differentiating the boundary condition with respect to t. For the tau method $\mathbf{v}$ consists of the spectral coefficients $\hat{v}_0,..., \hat{v}_N$ and L_N is upper triangular (except for the last row which comes from the boundary condition), while for the collocation method $\mathbf{v}$ consists of the physical space values $v_0,..., v_N$ and the matrix is full.

3. Time Differencing and Time Filtering

Finite differences in t are generally used to solve spectral equations such as (2.9). When using an explicit scheme one can take advantage of the spectral representation. Specifically, the derivative (2.5) may be computed using the recurrence relation $c_{n-1}\hat{v}_{n-1}^{(1)} - \hat{v}_{n+1}^{(1)} = 2n\hat{v}_n$ $(n = 1,..., N$, with $\hat{v}_N^{(1)} = \hat{v}_{N+1}^{(1)} = 0)$, and the transforms (2.7) and (2.8) in the collocation method may be computed using the FFT algorithm. Therefore, explicit schemes require at most $\mathcal{O}(N \log N)$ operations per time step. In contrast, when using an implicit scheme one must solve an equation of the form $L_N \mathbf{v} = \mathbf{f}$ for $\mathbf{v}$ at each time step. If this equation is solved directly the special properties of the spectral representation cannot in general be exploited, so such schemes require at least $\mathcal{O}(N^2)$ operations per time step.

A typical explicit scheme for the model problem (2.9) is the modified Euler scheme

$$\begin{aligned} \mathbf{v}^{(k+1/2)} &= \mathbf{v}^{(k)} + \tfrac{1}{2}\Delta t\, L_N \mathbf{v}^{(k)}, \\ \mathbf{v}^{(k+1)} &= \mathbf{v}^{(k)} + \Delta t\, L_N \mathbf{v}^{(k+1/2)}, \end{aligned} \tag{3.1}$$

(with the superscript k denoting values at time level $t_k = k\,\Delta t$) which may be written as

$$\mathbf{v}^{(k+1)} = K_N(\Delta t)\, \mathbf{v}^{(k)}, \tag{3.2}$$

where $K_N(\Delta t) = I + \Delta t\, L_N + \frac{1}{2}(\Delta t)^2 L_N^2$. It is shown in [1] that for the collocation method this scheme is *algebraically stable* with stability condition

$$N^2\, \Delta t \leqslant 8. \tag{3.3}$$

That is, if $[0, T]$ denotes the interval on which the solution is sought and $\|\cdot\|$ denotes any matrix norm, then there exist numbers $r \geqslant 0$ and $s \geqslant 0$ such that

$$\|[K_N(\Delta t)]^k\| = \mathcal{O}(N^{r+sk\,\Delta t}) \qquad (N \to \infty, \Delta t \to 0) \tag{3.4}$$

for all $k\,\Delta t \in [0, T]$, provided the limits are taken so that (3.3) is satisfied. This type of stability is important because it implies convergence (i.e., $\|u - v\| \to 0$ as $N \to \infty$, $\Delta t \to 0$) provided a corresponding consistency condition is satisfied [1].

The small time step $[\Delta t = \mathcal{O}(1/N^2)]$ required by (3.3) is typical of explicit schemes applied to Chebyshev spectral equations and is related to the increased resolution of the Chebyshev series (2.3) near the boundaries $x = \pm 1$. Gottlieb and Turkel [2] suggest that (3.3) may be relaxed by replacing the derivative calculation (2.5) by the filtered version

$$\Delta t\, \hat{v}_n^{(1)} = \frac{2\,\Delta t}{c_n} \sum_{\substack{p=n+1 \\ n+p \text{ odd}}}^{N} p f(p^2 \alpha\, \Delta t)\, \hat{v}_p, \tag{3.5}$$

where α is a constant and f is a suitably chosen filter function. It can be seen that this filter effectively expands the x-scale for Chebyshev mode n by the factor $1/f(n^2 \alpha\, \Delta t)$. By choosing the filter properly the column norm of the matrix representing the derivative in spectral space may be reduced from N^2 to $\mathcal{O}(1)$. Gottlieb and Turkel argue that therefore the stability condition (3.3) will be relaxed to $\Delta t = \mathcal{O}(1)$ so that "time steps are chosen by accuracy requirements alone."

4. Absolute Stability Analysis

The stability defined in the previous section is important for convergence theory, but in practice a different type of stability is needed. Since the true solution does not grow with time it is reasonable to require that the approximate solution not grow in time for a fixed time step Δt. This will be the case in general if and only if

$$\rho[K_N(\Delta t)] \leqslant 1, \tag{4.1}$$

where $\rho(A)$ denotes the spectral radius of a matrix A. If (4.1) holds we say that the scheme is *absolutely stable*. We find numerically that (4.1) holds for the unfiltered method with the stability condition

$$\beta = N^2 \, \Delta t \leqslant \beta^*, \tag{4.2}$$

where β^* is shown in Fig. 1 for $1 \leqslant N \leqslant 64$. This condition is similar to condition (3.3) for algebraic stability, even though absolute stability is the stronger requirement. The claim of unconditional stability in the sense (3.4) for the filtered method therefore might be interpreted as also implying unconditional stability in the sense (4.1). Therefore, we consider below the effect of the filter on absolute stability for the model problem. Two choices of the filter function are considered and results are presented for the collocation method only as the tau results are similar.

4.1. *Trigonometric Filter*

The trigonometric filter function in [2] is

$$f(z) = \frac{8 \sin(z) - \sin(2z)}{6z}, \tag{4.3}$$

as shown in Fig. 2. The corresponding spectral radius $\rho[K_N(\Delta t)]$ is shown in Fig. 3 for $N = 16$ in terms of the normalized time step $\beta = N^2 \, \Delta t$ and the normalized filter

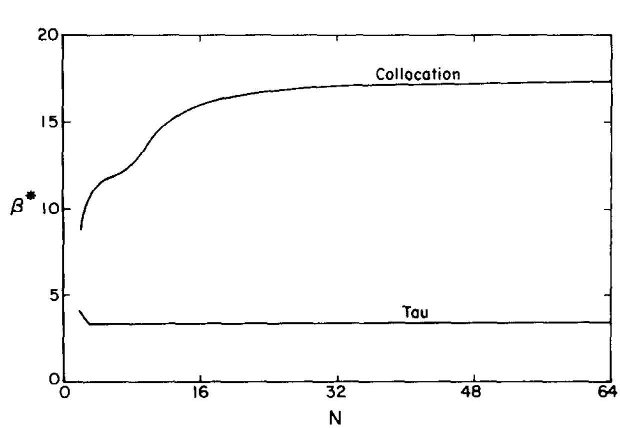

FIG. 1. Normalized time step β^* at which the Chebyshev-tau and Chebyshev-collocation methods become absolutely unstable [see (4.2)].

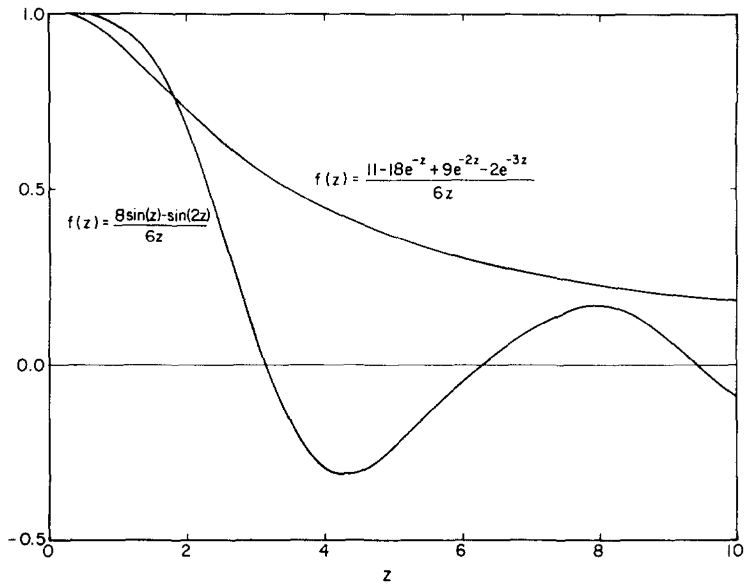

FIG. 2. Trigonometric and exponential filter functions.

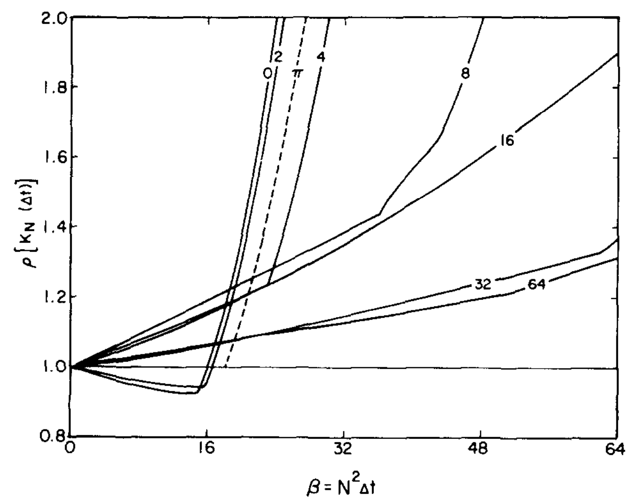

FIG. 3. Spectral radius of the evolution matrix $K_N(\Delta t)$ as a function of the normalized time step $\beta = N^2 \Delta t$ for various values of the normalized filter parameter $\gamma = N^2 \alpha \Delta t$ as labelled, for the collocation method with $N = 16$ and using the trigonometric filter (4.3).

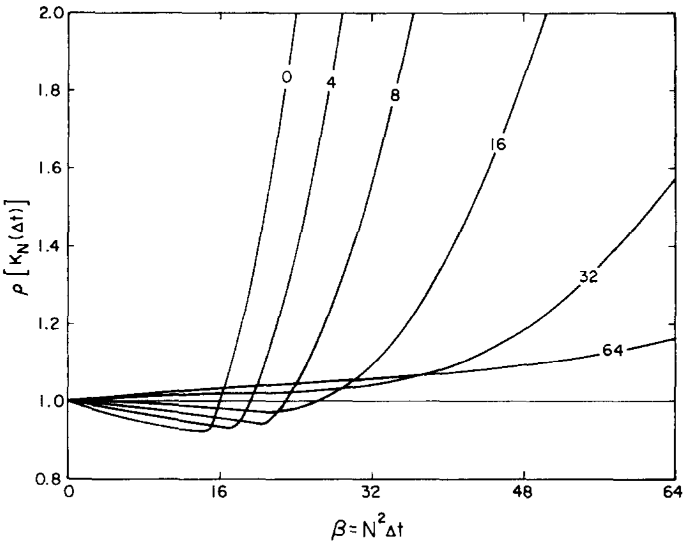

FIG. 4. Spectral radius of the evolution matrix $K_N(\Delta t)$ as a function of the normalized time step $\beta = N^2\,\Delta t$ for various values of the normalized filter parameter $\gamma = N^2\alpha\,\Delta t$ as labelled, for the collocation method with $N = 16$ and using the exponential filter (4.4).

parameter $\gamma = N^2\alpha\,\Delta t$; with this scaling the results are to a good approximation independent of N. It can be seen that for small filtering ($\gamma < \pi$) the stability properties of the scheme are not changed substantially, while for large filtering ($\gamma > \pi$) the scheme is rendered absolutely *unstable* for all Δt. The instability for $\gamma > \pi$ might be expected, as in this case some of the filter coefficients $f(n^2\alpha\,\Delta t)$ may be negative; since this changes the sign of the x-derivative for the corresponding Chebyshev modes, the problem may then become ill-posed with the boundary condition applied at $x = -1$.

4.2. *Exponential Filter*

The exponential filter function suggested in [4] is

$$f(z) = \frac{1}{6z}\,[11 - 18e^{-z} + 9e^{-2z} - 2e^{-3z}] \tag{4.4}$$

(see Fig. 2). The corresponding spectral radius $\rho[K_N(\Delta t)]$ is shown in Fig. 4 for $N = 16$; the results are similar to those obtained with the trigonometric filter. To investigate whether in some cases where $\rho[K_N(\Delta t)] > 1$ the amplification may be small enough that the filter is still useful, we consider the error obtained for the initial condition

$$u(x, 0) = e^{-[(x+0.5)/0.2]^2}. \tag{4.5}$$

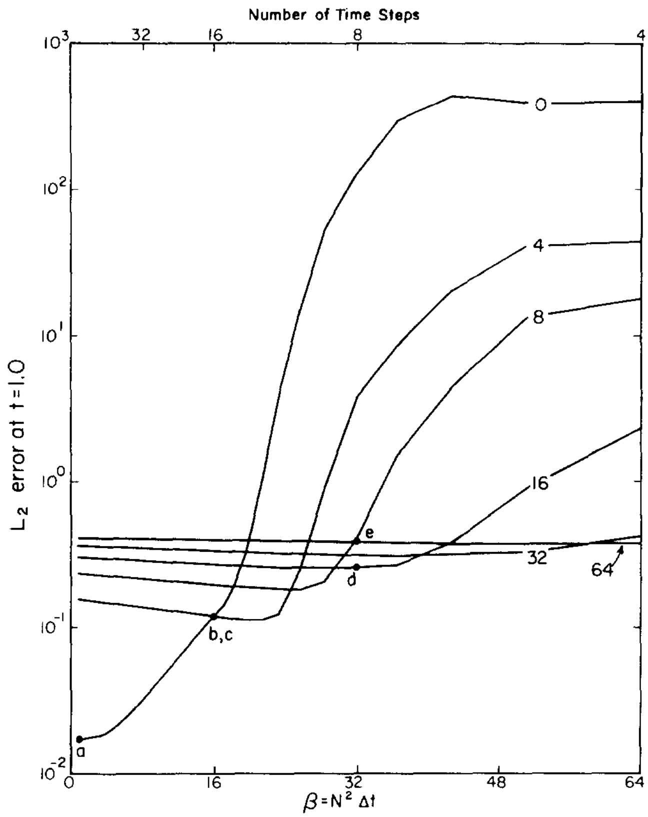

FIG. 5. Continuous L_2 error at $t = 1.0$ as a function of the normalized time step $\beta = N^2 \Delta t$ for the collocation method with $N = 16$, initial condition (4.5) and filtering $\gamma = N^2 \alpha \Delta t$ as labelled. The points a through e correspond to the solutions shown in Figs. 6 and 7.

The continuous L_2 error (computed by Romberg integration and scaled by the domain length 2) at $t = 1$ is shown in Fig. 5 as a function of β for various values of γ. As the amount of filtering increases the error for large β decreases, so that for $\gamma = 32$ and $\gamma = 64$ the error is reasonably small even though $\rho[K_N(\Delta t)] > 1$. Thus in a practical sense the filter can in fact stabilize the calculation.

To see whether these results are useful we present the corresponding solutions. The unfiltered solution ($\gamma = 0$) at $t = 1$ is shown in Fig. 6 for $\beta = 1$ (essentially no time differencing error) and for $\beta = 16$ (just below the stability criterion) along with the analytical solution for comparison. Filtered results are shown in Fig. 7 for various values of β and γ as indicated in Fig. 5. The net effect of the filter is to concentrate the oscillatory part of the solution near $x = -1$ and the smooth part near $x = +1$.

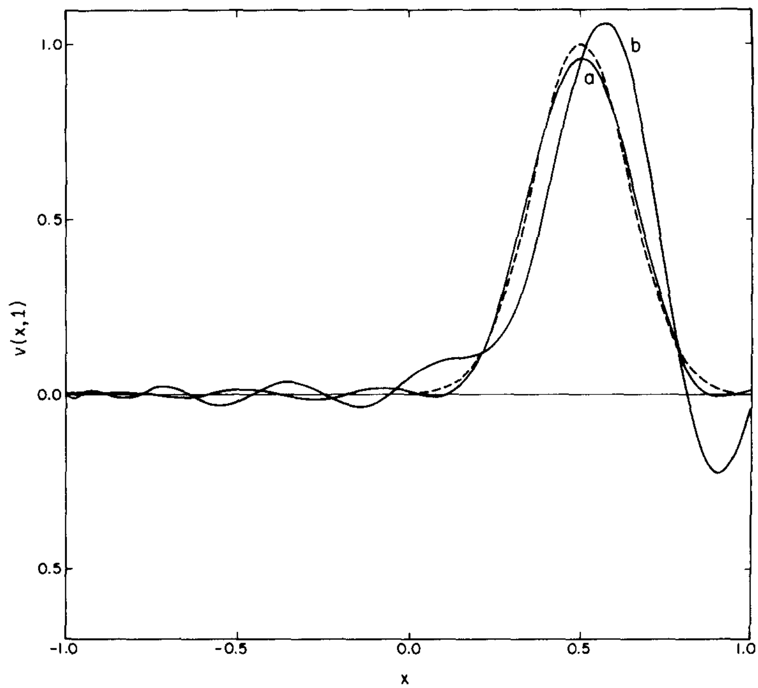

FIG. 6. Unfiltered solutions ($\gamma = 0$) at $t = 1.0$ as for Fig. 5, with (a) $\beta = 1$ and (b) $\beta = 16$. The dashed curve is the analytical solution.

This effect becomes more pronounced as the amount of filtering γ increases and is similar to the computational dispersion seen in centered finite difference approximations to this problem.

5. CONCLUDING REMARKS

We have examined the Gottlieb–Turkel time filter for Chebyshev spectral methods in some detail. This filter is claimed to produce unconditional (algebraic) stability so that "time steps are chosen by accuracy requirements alone." This stability is useful for convergence theory (in the limit as $N \to \infty$ and $\Delta t \to 0$) but absolute stability is needed in practice. Our results indicate that

(1) small amounts of filtering do not change the absolute stability properties significantly;

(2) large amounts of filtering render the scheme absolutely *unstable* for any time step;

(3) the filter results in computational dispersion similar to that seen in some finite difference schemes.

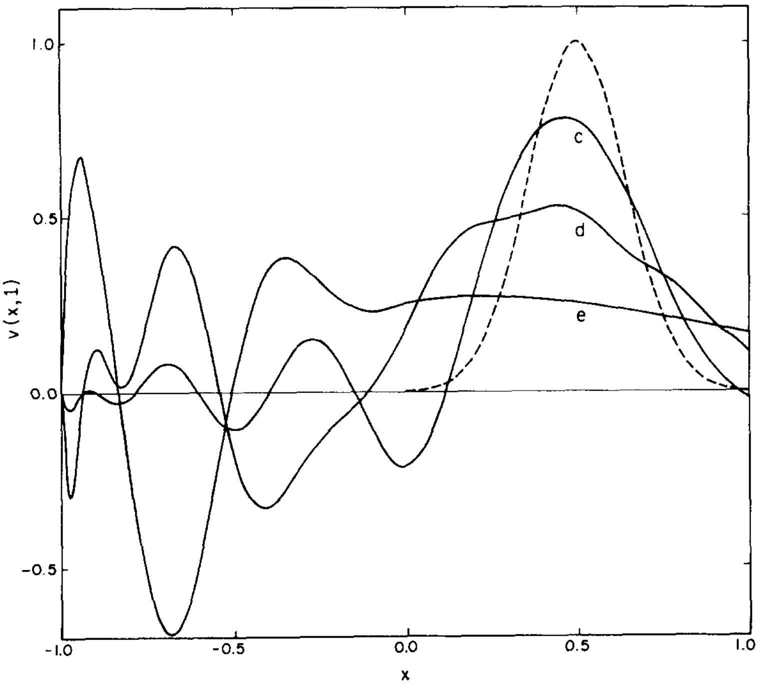

FIG. 7. Filtered solutions at $t = 1.0$ as for Fig. 5, with (c) $\beta = 16$, $\gamma = 4$, (d) $\beta = 32$, $\gamma = 16$ and (e) $\beta = 32$, $\gamma = 64$. The dashed curve is the analytical solution.

In physical problems more complicated than the one-dimensional advection equation considered here, several modes with different wave or advective speeds may be present. The time filter could be useful in such cases if it could stabilize the fastest modes without distorting the slower modes of primary interest. However, unless the various modes can be treated independently, the same filter parameter α and time step Δt must be used in computing all spatial derivatives and hence the amount of filtering (as measured by the normalized filter parameter γ) will be the same for each mode. The curves for constant γ in Fig. 5 suggest that if γ is large enough to stabilize the fast modes (for which β is large) then the slow modes (for which β is small) will be significantly distorted. This conclusion remains to be verified in actual practice.

ACKNOWLEDGMENTS

We wish to acknowledge helpful discussions with Professors David Gottlieb and Eli Turkel. This work was supported by the Office of Naval Research under Contracts N00014-83-K-0068 and N00014-C-79-0793 and the National Science Foundation under Grant ATM-82-07563.

REFERENCES

1. D. GOTTLIEB AND S. A. ORSZAG, "Numerical Analysis of Spectral Methods," NSF-CBMS Monograph No. 26, Soc. Ind. and Appl. Math., Philadelphia, 1977.
2. D. GOTTLIEB AND E. TURKEL, *Stud. Appl. Math.* **63** (1980), 67.
3. S. A. ORSZAG, *J. Comput. Phys.* **37** (1980), 70.
4. S. A. ORSZAG AND D. GOTTLIEB, High resolution spectral calculations of inviscid compressible flows, *in* "Approximation Methods for Navier-Stokes Problems" (R. Rautmann, Ed.), Springer-Verlag, Berlin, 1980.